Fort Street High School

2022 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:____________________________ **TEACHER:**____________________________

STUDENT NUMBER:

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/16	/16	/15	/14	/14	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 What is the unit vector in the same direction as $\underset{\sim}{a} = 2i + j - 3k$?

A. $\dfrac{-1}{14}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

B. $\dfrac{-1}{14}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

C. $\dfrac{-1}{\sqrt{14}}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

D. $\dfrac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

2 Given that $z^n = a + bi$ and $|z| = 1$ where a and b are real, which of the following is equal to z^{-n} ?

A. $a + bi$

B. $a - bi$

C. $-a + bi$

D. $-a - bi$

3 Consider the statement below:

"If z is even, then x and y are either both even or both odd."

Which of the following is equivalent to the above statement?

A. If x and y are either both even or both odd, then z is even.

B. If z is odd, then exactly one of x and y is even.

C. If x and y are either both even or both odd, then z is odd.

D. If exactly one of x and y is even, then z is odd.

4 Consider the vectors $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{OB} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{OC} = \begin{bmatrix} 3 \\ \mu \\ \lambda \end{bmatrix}$, where O is the origin.

What are the values of μ and λ if the points A, B and C are collinear?

A. $\mu = 3, \quad \lambda = 2$

B. $\mu = -3, \quad \lambda = 2$

C. $\mu = 3, \quad \lambda = -2$

D. $\mu = -3, \quad \lambda = -2$

5 One of the roots of the quadratic equation $iz^2 + 3z + 3 - 11i = 0$ is $3 + 2i$.

What is the other root?

A. $-2i$

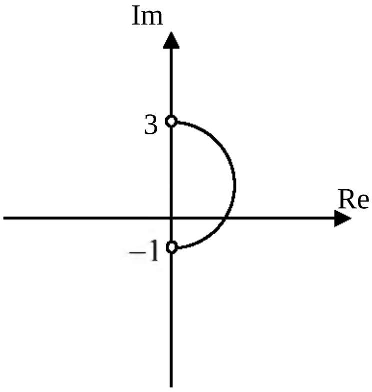
B. $-3 + i$

C. $3 - 2i$

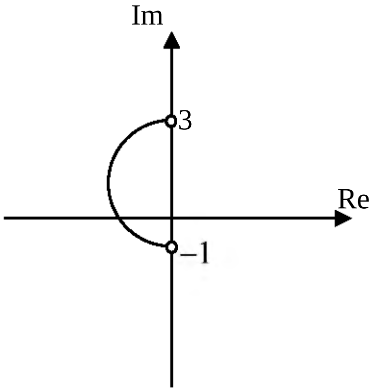
D. $-5 - 3i$

6 Which of the following is the locus of z such that $\arg(z+3i)-\arg(z-i)=\dfrac{\pi}{2}$?

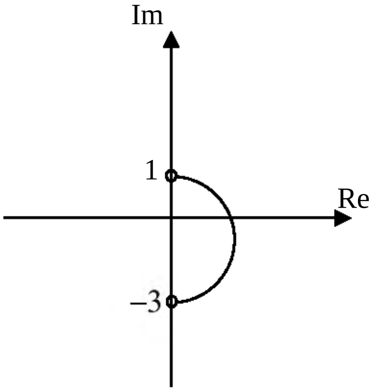
A.



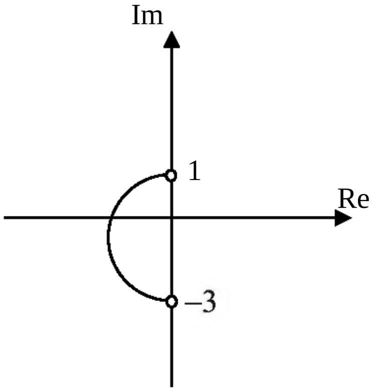
B.



C.



D.



7 Which of the following definite integrals has a positive value?

A. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} x^3(1+\cos x)\,dx$

B. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\sin^{-1}x}{1+x^4}\right)dx$

C. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\cos^{-1}x}{e^{-x^2}}\right)dx$

D. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\tan^{-1}x}{e^x+e^{-x}}\right)dx$

8 Given that z is a complex number such that $\operatorname{Re}(z) \neq 0$, which of the following is equivalent to $\dfrac{4z\bar{z}}{(z+\bar{z})^2}$?

A. $1+\left(\dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)^2$

B. $4\left(\operatorname{Im}(z) \times \operatorname{Re}(z)\right)$

C. $4\left(1+\left[\operatorname{Im}(z)+\operatorname{Re}(z)\right]^2\right)$

D. $\dfrac{2 \times \operatorname{Im}(z)}{\left[\operatorname{Re}(z)\right]^2}$

9 Which of the following is the negation of the statement:

"$\forall p \in P$ p is of the form $4m+1 \Rightarrow p$ can be written as a sum of two squares"?

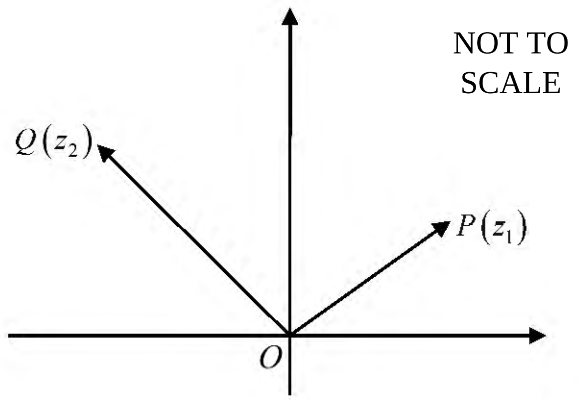
A. $\forall p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

B. $\exists p \in P$, p is not of the form $4m+1$ and p can be written as a sum of two squares.

C. $\forall p \in P$, p is not of the form $4m+1$ or p cannot be written as a sum of two squares.

D. $\exists p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

10 In the diagram the vectors OP and OQ represent the complex numbers z_1 and z_2 respectively.



If $\dfrac{z_2}{z_1} = \sqrt{3}\,i$, what is the value of $\left|\dfrac{z_1 + z_2}{z_2 - z_1}\right|$?

A. 1

B. $\sqrt{3}$

C. $\dfrac{1}{\sqrt{3}}$

D. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks**)** Use a SEPARATE writing booklet

(a) If $z=-\sqrt{3}+i$ and $w=1+i$:

(i) Find $\text{Arg}(zw)$ in terms of π. **2**

(ii) Write $\dfrac{z}{w}$ in the form $x+iy$, where x and y are real numbers. **2**

(b) Sketch the region on the Argand diagram where: **2**

$$\arg(z+1-2i)=\arg(z-3+i).$$

(c) Find the vector equation for the sphere $x^2+4x+y^2-6y+z^2=12$. **2**

(d) Point A has position vector $8\underset{\sim}{i}+13\underset{\sim}{j}-2\underset{\sim}{k}$ the point B has position vector $10\underset{\sim}{i}+14\underset{\sim}{j}-4\underset{\sim}{k}$ and the point C has position vector $9\underset{\sim}{i}+9\underset{\sim}{j}+6\underset{\sim}{k}$.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Fin|. $\left|\overrightarrow{CB}\right|$. **2**

(iii) What is the size of the acute angle between $\overrightarrow{AB}$ and $\overrightarrow{CB}$? Answer correct to the nearest degree. **3**

(e) Find $\displaystyle\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$. **2**

End of Question 11

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) (i) Find A and B such that $\dfrac{15}{x^2-2x-15}=\dfrac{A}{x+3}+\dfrac{B}{x-5}$. **2**

(ii) Hence or otherwise find $\displaystyle\int \frac{x^2-2x}{x^2-2x-15}\,dx$. **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\tan^2 x\,dx$. **3**

(c) Use proof by contrapositive to prove that: **3**

If a^2-2a+7 is even then a is odd.

(d) Solve $\left|e^{i\theta}+\sqrt{2}\right|=1$. **3**

(e) Use the substitution $x=2\sin\theta$ to find $\displaystyle\int \frac{dx}{\left(4-x^2\right)^{\frac{3}{2}}}$. **3**

End of Question 12

Question 13 (14 marks) Use a SEPARATE writing booklet

(a) Find $\int \cos^{-1} x \, dx$ **3**

(b) Use the substitution $t = \tan \frac{x}{2}$ or otherwise to evaluate: **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5} \, dx.$$

(c) The line l_1 passes through the points $A(2,-1,4)$ and $B(4,-3,2)$.

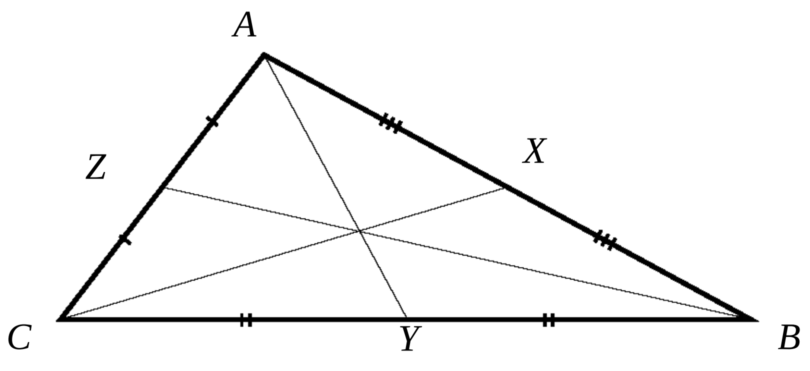
(i) Show that a vector equation of line l_1 is $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. **2**

(ii) The equation of line l_2 is given by $\frac{x+3}{k} = \frac{4-y}{6} = z - 1$. **3**

Find the value of k for which l_1 and l_2 intersect.

(d) The diagram shows a triangle *ABC*.

The points *X*, *Y* and *Z* bisect the intervals *AB*, *BC* and *CA* respectively.



Show that $\overrightarrow{AY} + \overrightarrow{BZ} + \overrightarrow{CX} = 0$ **3**

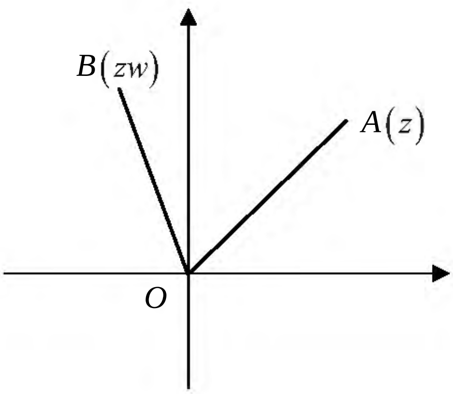
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) If $x \le 1$ and $y \ge 1$ show that $x + y \ge 1 + xy$. **2**

(b) Simplify fully $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2}$, given that ω is a non-real cube root of unity. **2**

(c) As shown on the Argand diagram below, the complex numbers *z* and *zw* are represented by the points *A* and *B* respectively.



Given $z = re^{i\theta}$ and $w = e^{i\frac{\pi}{3}}$, where $r > 0$,

(i) Explain why *OAB* is an equilateral triangle. **2**

(ii) Write the complex number $z - zw$ in exponential form in terms of *r* and θ. **2**

(d) (i) Use a suitable substitution to show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x + \sin x}\,dx$. **3**

(e) A recurrence relation is defined by $u_{n+1} = \dfrac{3u_n - 1}{4u_n - 1}$ and $u_1 = 1$. **3**

Use Mathematical Induction to prove that $u_n = \dfrac{n}{2n-1}$ for $n \ge 1$.

End of Question 14

Question 15 (14 marks) Use a SEPARATE writing booklet

(a) (i) Show that $a^2 + 9b^2 \geq 6ab$, where a and b are real numbers. **1**

(ii) Hence, or otherwise, show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. **2**

(iii) Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$. **2**

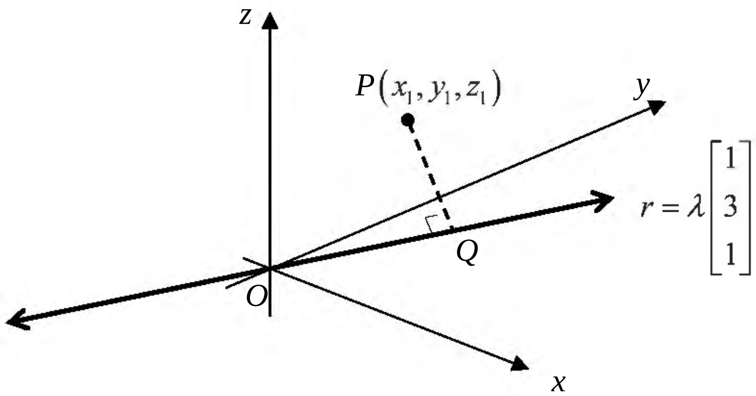
(b) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$. **4**

Use integration by parts to show that $I_n = \frac{2n}{2n+1} I_{n-1}$.

Question 15 Continues on Page 13

(c) The diagram below shows the line with vector equation $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

The point $P(x_1, y_1, z_1)$ is any point in the three-dimensional space.
Q is the point on $\underset{\sim}{r}$ such that PQ is a minimum.

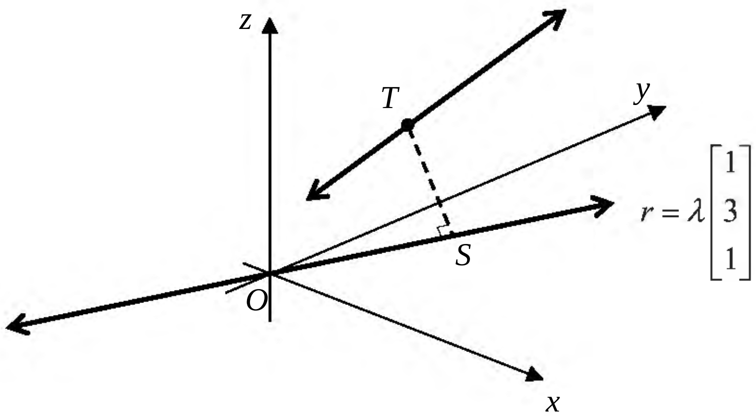


(i) Using vector projection methods show that: **2**

$$\overrightarrow{OQ} = \frac{x_1 + 3y_1 + z_1}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

It is given that line $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ does not intersect $\underset{\sim}{r}$.

Point T is on the line $\underset{\sim}{c}$ and S is a point on $\underset{\sim}{r}$ such that TS is a minimum.



(ii) By first expressing $\underset{\sim}{c}$ in parametric form, find an expression for the vector $\overrightarrow{TS}$ in terms of t. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) (i) Prove that: **2**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\frac{\theta}{2}.$$

(ii) Find the roots of the equation $w^5 = 1$. **2**
Express your answers in the form $(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$.

(iii) Using parts (i) and (ii) find the roots of the equation: **3**

$$\left(\frac{2+z}{2-z}\right)^5 = 1$$

(b) (i) Show that $\dfrac{1}{(k-1)k(k+1)} \geq \dfrac{1}{k^3}$ where k is a positive integer greater than 1. **2**

(ii) Given that: **3**

$$S_n = \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots + \frac{1}{n^3}$$ where n is an integer and $n \geq 3$.

Using part (i) or otherwise, prove that $S_n < \dfrac{1}{12}$.

(c) Find $\displaystyle\int \frac{\ln x - 1}{[x + \ln x]^2}\,dx$. **3**

End of paper

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 What is the unit vector in the same direction as $\underset{\sim}{a} = 2i + j - 3k$?

A. $\dfrac{-1}{14}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

B. $\dfrac{-1}{14}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

C. $\dfrac{-1}{\sqrt{14}}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

D. $\dfrac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

2 Given that $z^n = a + bi$ and $|z| = 1$ where a and b are real,

which of the following is equal to z^{-n} ?

A. $a + bi$

B. $a - bi$

C. $-a + bi$

D. $-a - bi$

3 Consider the statement below:

"If z is even, then x and y are either both even or both odd."

Which of the following is equivalent to the above statement?

A. If x and y are either both even or both odd, then z is even.

B. If z is odd, then exactly one of x and y is even.

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D. If exactly one of x and y is even, then z is odd.

4 Consider the vectors $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{OB} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{OC} = \begin{bmatrix} 3 \\ \mu \\ \lambda \end{bmatrix}$, where O is the origin.

What are the values of μ and λ if the points A, B and C are collinear?

A. $\mu = 3, \quad \lambda = 2$

B. $\mu = -3, \quad \lambda = 2$

C. $\mu = 3, \quad \lambda = -2$

D. $\mu = -3, \quad \lambda = -2$

5 One of the roots of the quadratic equation $iz^2 + 3z + 3 - 11i = 0$ is $3 + 2i$.

What is the other root?

A. $-2i$

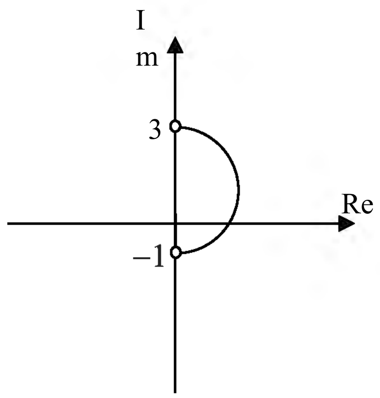
B. $-3 + i$

C. $3 - 2i$

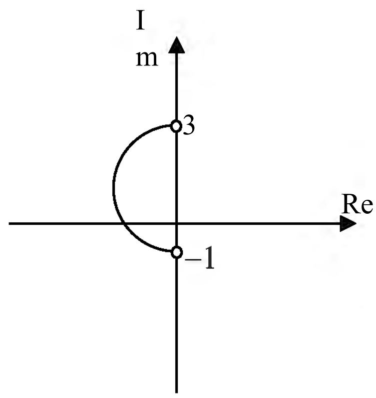
D. $-5 - 3i$

6 Which of the following is the locus of z such that $\arg(z+3i)-\arg(z-i)=\frac{\pi}{2}$?

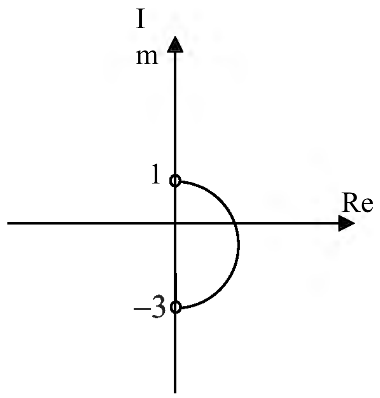
A.



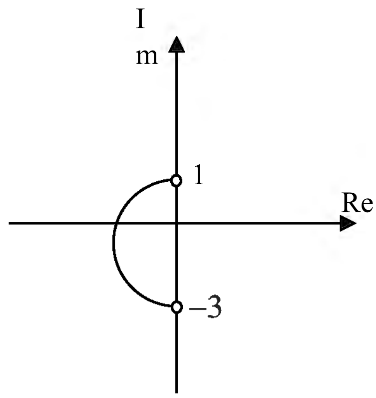
B.



C.



D.



7 Which of the following definite integrals has a positive value?

A. $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^3(1+\cos x)\,dx$

B. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\sin^{-1} x}{1+x^4}\right) dx$

C. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\cos^{-1} x}{e^{-x^2}}\right) dx$

D. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\tan^{-1} x}{e^{x}+e^{-x}}\right) dx$

8 Given that z is a complex number such that $\operatorname{Re}(z) \neq 0$, which of the following is equivalent to $\dfrac{4z\bar{z}}{(z+\bar{z})^2}$?

A. $1+\left(\dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)^2$

B. $4(\operatorname{Im}(z)\times\operatorname{Re}(z))$

C. $4\left(1+[\operatorname{Im}(z)+\operatorname{Re}(z)]^2\right)$

D. $\dfrac{2\times\operatorname{Im}(z)}{[\operatorname{Re}(z)]^2}$

9 Which of the following is the negation of the statement:

"$\forall p \in P$ p is of the form $4m+1 \Rightarrow p$ can be written as a sum of two squares"?

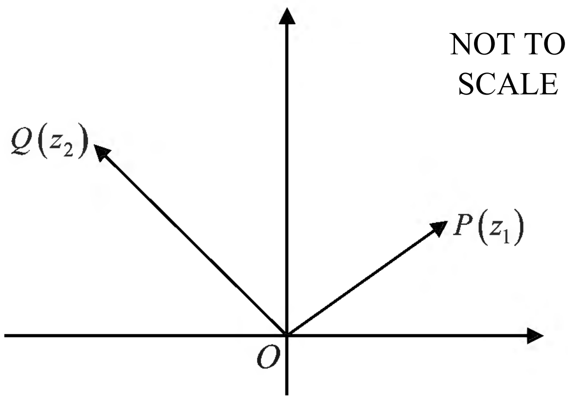
A. $\forall p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

B. $\exists p \in P$, p is not of the form $4m+1$ and p can be written as a sum of two squares.

C. $\forall p \in P$, p is not of the form $4m+1$ or p cannot be written as a sum of two squares.

D. $\exists p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

10 In the diagram the vectors OP and OQ represent the complex numbers z_1 and z_2 respectively.



If $\dfrac{z_2}{z_1} = \sqrt{3}\,i$, what is the value of $\left|\dfrac{z_1 + z_2}{z_2 - z_1}\right|$?

A. 1

B. $\sqrt{3}$

C. $\dfrac{1}{\sqrt{3}}$

D. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a SEPARATE writing booklet

(a) If $z = -\sqrt{3} + i$ and $w = 1 + i$:

$\arg z = \frac{5\pi}{6}$ and $\arg w = \frac{\pi}{4}$

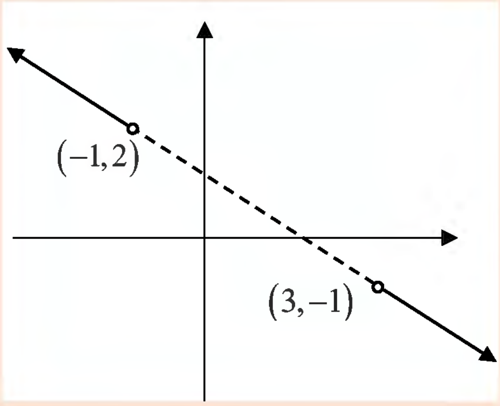
$$\begin{aligned} \operatorname{Arg}(zw) &= \arg z + \arg w \\ &= \frac{5\pi}{6} + \frac{\pi}{4} \\ &= \frac{13\pi}{12} \\ \operatorname{Arg}(zw) &= -\frac{11\pi}{12} \end{aligned}$$

(i) Find $\operatorname{Arg}(zw)$ in terms of π. **2**

$$\begin{aligned} \frac{z}{w} &= \frac{-\sqrt{3} + i}{1 + i} \\ &= \frac{-\sqrt{3} + i}{1 + i} \times \frac{1 - i}{1 - i} \\ &= \frac{-\sqrt{3} + i + i\sqrt{3} + 1}{\sqrt{2}} \\ &= \frac{1 - \sqrt{3}}{\sqrt{2}} + i\frac{1 + \sqrt{3}}{\sqrt{2}} \end{aligned}$$

(ii) Write $\frac{z}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Sketch the region on the Argand diagram where: **2**



$\arg(z+1-2i)=\arg(z-3+i)$.

$(x^2+4x+4)+(y^2-6y+9)+z^2=12+4+9$

$(x+2)^2+(y-3)^2+z^2=25$

Centre is $(-2,3,0)$ Radius is 5.

Vector equation is $\left|\underset{\sim}{r}-\begin{bmatrix}-2\\3\\0\end{bmatrix}\right|=5$

(c) Find the vector equation for the sphere $x^2+4x+y^2-6y+z^2=12$. **2**

(d) Point A has position vector $8\underset{\sim}{i}+13\underset{\sim}{j}-2\underset{\sim}{k}$ the point B has position vector $10\underset{\sim}{i}+14\underset{\sim}{j}-4\underset{\sim}{k}$ and the point C has position vector $9\underset{\sim}{i}+9\underset{\sim}{j}+6\underset{\sim}{k}$.

$\overrightarrow{AB}=\begin{bmatrix}10\\14\\-4\end{bmatrix}-\begin{bmatrix}8\\13\\-2\end{bmatrix}=\begin{bmatrix}2\\1\\-2\end{bmatrix}$

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $\left|\overrightarrow{CB}\right|$. **2**

$$\left|\overrightarrow{CB}\right| = \sqrt{(9-10)^2+(9-14)^2+(6+4)^2}$$
$$= \sqrt{1+25+100}$$
$$= \sqrt{126} = 3\sqrt{14}$$

(iii) What is the size of the acute angle between $\overrightarrow{AB}$ and $\overrightarrow{CB}$? **3**

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad \text{Part i)} \quad \overrightarrow{BC} = \begin{bmatrix} 10-9 \\ 14-9 \\ -4-(-6) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -10 \end{bmatrix}$$

$$\cos\theta = \frac{\overrightarrow{AB}\bullet\overrightarrow{CB}}{\left|AB\right|\left|CB\right|}$$
$$\frac{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\bullet\begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}}{\sqrt{2^2+1^2+(-2)^2}\times 3\sqrt{14}}$$
$$= \frac{2+5+20}{3\times 3\sqrt{14}}$$
$$= \frac{3}{\sqrt{14}}$$
$$\theta = \cos^{-1}\left(\tfrac{3}{\sqrt{14}}\right)$$
$$\theta \approx 37°$$

Answer correct to the nearest degree.

(e) Find $\displaystyle\int \frac{1}{\sqrt{5+4x-x^2}}\,dx$. **2**

$$\int \frac{1}{\sqrt{5+4x-x^2}}\,dx = \int \frac{1}{\sqrt{5-\left(x^2-4x\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{5+4-\left(x^2-4x+4\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2-\left(x-2\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x-2}{3}\right)+C$$

End of Question 11

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) (i) Find A and B such that $\dfrac{15}{x^2-2x-15}=\dfrac{A}{x+3}+\dfrac{B}{x-5}$. **2**

$$\frac{15}{x^2-2x-15}=\frac{A}{x+3}+\frac{B}{x-5}$$

$$15=A(x-5)+B(x+3)$$

Let $x=5$

$$15=+B(5+3)$$

$$\frac{15}{8}=B$$

Let $x=-3$

$$15=A(-3-5)$$

$$-\frac{15}{8}=A$$

$$\frac{15}{x^2-2x-15}=-\frac{15}{8(x+3)}+\frac{15}{8(x-5)}$$

1 – correct answer for A

1 – correct answer for B

(ii) Hence or otherwise find $\displaystyle\int\frac{x^2-2x}{x^2-2x-15}\,dx$. **2**

$$\int\frac{x^2-2x}{x^2-2x-15}\,dx=\int\frac{x^2-2x-15+15}{x^2-2x-15}\,dx$$

$$=\int\frac{x^2-2x+15}{x^2-2x-15}\,dx+\int\frac{15}{x^2-2x-15}\,dx$$

$$=\int dx+\int\frac{15}{x^2-2x-15}\,dx$$

$$=\int dx-\frac{15}{8}\int\frac{1}{x+3}dx+\frac{15}{8}\int\frac{1}{x-5}dx$$

$$=x+\frac{15}{8}\left[\ln(x-5)-\ln(x+3)\right]+C$$

$$=x+\frac{15}{8}\ln\left(\frac{x-5}{x+3}\right)+C$$

1 – correct expansion or equivalent merit

1 – correct integral

1 – correct answer

(b) Evaluate $\int_{0}^{\frac{\pi}{4}} \sec^4 x \tan^2 x\, dx$. **3**

$$\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x dx = \int_0^{\frac{\pi}{4}} \sec^2 x \sec^2 x \tan^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x \left(1 + \tan^2 x\right) \tan^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x \left(\tan^2 x + \tan^4 x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x dx + \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x dx$$

$$= \left[\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5}\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\tan^3 \frac{\pi}{4}}{3} + \frac{\tan^5 \frac{\pi}{4}}{5}\right) - \left(\frac{\tan^3 0}{3} + \frac{\tan^5 0}{5}\right)$$

$$= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

1 – correct expansion or equivalent merit

1 – correct integral

1 – correct answer

Alternative Solution

$$\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \sec^2 x \times \sec^2 x \times \tan^2 x\, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(1 + \tan^2 x\right) \times \tan^2 x \times \sec^2 x\, dx$$

Let $u = \tan x$ when $x = \frac{\pi}{4}$ $u = 1$

$du = \sec^2 x\, dx$ when $x = 0$ $u = 0$

$$\int_0^1 \left(1 + u^2\right) \times u^2\, du = \left[\frac{u^3}{3} + \frac{u^5}{5}\right]_0^1$$

$$= \frac{8}{15}$$

(c) Use proof by contrapositive to prove that: **3**

If $a^2 - 2a + 7$ is even then a is odd.

Contrapositive is:

If a is not odd then $a^2 - 2a + 7$ is not even.

Which is equivalent to

If a is even then $a^2 - 2a + 7$ is odd.

Let $a = 2k$ where k is an integer.

Then :

$$a^2 - 2a + 7 = (2k)^2 - 2(2k) + 7$$

$$= 2(2k^2 - 2k + 3) + 1$$

$2k^2 - 2k + 3$ is an integer as k is an integer

$$\therefore 2(2k^2 - 2k + 3) + 1 \text{ is odd}$$

Thus If a is even then $a^2 - 2a + 7$ is odd.

Thus If $a^2 - 2a + 7$ is even then a is odd.

1 – correct contrapositive statement

1 – defining k as an integer or equivalent merit

1 – qualifying $2k^2 - 2k + 3$ as an integer and $2(2k^2 - 2k + 3) + 1$ as odd

(d) Solve $\left| e^{i\theta} + \sqrt{2} \right| = 1$. **3**

Algebraic Solution

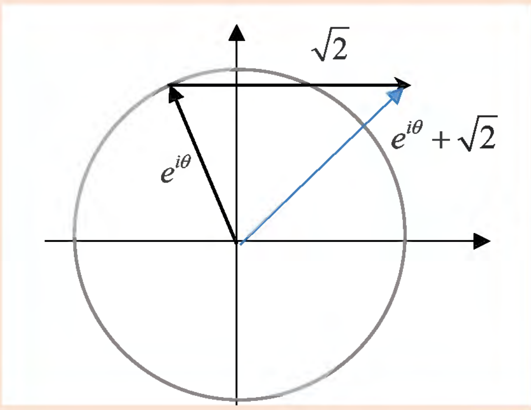
$$\left| e^{i\theta} + \sqrt{2} \right| = 1$$
$$\left| \cos\theta + i\sin\theta + \sqrt{2} \right| = 1$$
$$\left(\cos\theta + \sqrt{2} \right)^2 + (\sin\theta)^2 = 1$$
$$\cos^2\theta + 2\sqrt{2}\cos\theta + 2 + \sin^2\theta = 1$$
$$2\sqrt{2}\cos\theta + 3 = 1$$
$$2\sqrt{2}\cos\theta = -2$$
$$\cos\theta = -\frac{1}{\sqrt{2}} \qquad \text{for } -\pi < \theta \le \pi$$
$$\theta = \frac{3\pi}{4} \qquad or \qquad \theta = -\frac{3\pi}{4}$$

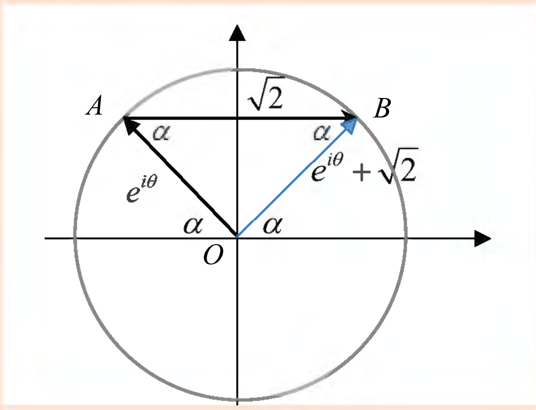
1 – correct calculation of magnitude

2 – correct answer for both angles

Geometric Solution

$e^{i\theta}$ *is a vector with endpoint on the unit circle. And* $\sqrt{2}$ *is a horizontal vector of length* $\sqrt{2}$





But $\left| e^{i\theta} + \sqrt{2} \right| = 1$ and so $e^{i\theta} + \sqrt{2}$ must have its endpoint on the unit circle.

Triangle OAB is isosceles right-angle triangle. $\alpha = \frac{\pi}{4}$ and so $\theta = \frac{3\pi}{4}$

A could also be in the third quadrant and so $\theta = \frac{-3\pi}{4}$

(e) Use the substitution $x = 2\sin\theta$ to find $\displaystyle\int \frac{dx}{\left(4-x^2\right)^{\frac{3}{2}}}$. **3**

$$\int \frac{dx}{\left(4-x^2\right)^{\frac{3}{2}}} = \int \frac{2\cos\theta}{\sqrt{\left(4-4\sin^2\theta\right)^3}}\,d\theta$$

$$= \int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{4}\tan\theta + C$$

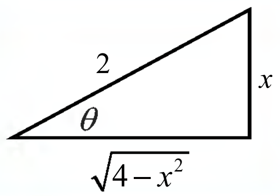
$$= \frac{x}{4\sqrt{4-x^2}} + C$$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\frac{x}{2} = \sin\theta$

$\tan\theta = \frac{x}{\sqrt{4-x^2}}$



1 – correct use of substitution

1 – correct integral in terms of θ

1 – correct integral in terms of x

End of Question 12

Question 13 (14 marks) Use a SEPARATE writing booklet

(a) Find $\int \cos^{-1} x \, dx$ **3**

$$I = \int \cos^{-1} x \, dx$$

$u = \cos^{-1} x \qquad v' = 1$

$u' = \dfrac{-1}{\sqrt{1-x^2}} \qquad v = x$

$$I = x\cos^{-1} x - \int x\left(\frac{-1}{\sqrt{1-x^2}}\right) dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int\left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int\left(-2x\left(1-x^2\right)^{-\frac{1}{2}}\right) dx$$

$$= x\cos^{-1} x - \frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

1 – application of partial integration

1 – working towards the final solution

1 – final answer

(b) Use the substitution $t = \tan\frac{x}{2}$ or otherwise to evaluate: **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx.$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx = \int_0^1 \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{6t - 4\left(1-t^2\right) + 5\left(1+t^2\right)}\,dt$$

$$= \int_0^1 \frac{2}{6t + 1 + 9t^2}\,dt$$

$$= \frac{2}{9}\int_0^1 \frac{1}{t^2 + \frac{2}{3}t + \frac{1}{9}}\,dt$$

$$= \frac{2}{9}\int_0^1 \frac{1}{\left(t + \frac{1}{3}\right)^2}\,dt$$

$$= \frac{2}{9}\int_0^1 \left(t + \tfrac{1}{3}\right)^{-2}\,dt$$

$$= \frac{2}{9}\left[-\left(t + \tfrac{1}{3}\right)^{-1}\right]_0^1$$

$$= -\frac{2}{9}\left[\left(1 + \tfrac{1}{3}\right)^{-1} - \left(0 + \tfrac{1}{3}\right)^{-1}\right]$$

$$= -\frac{2}{9}\left[\frac{3}{4} - 3\right]$$

$$= -\frac{2}{9}\left[-\frac{9}{4}\right]$$

$$= \frac{1}{2}$$

1 – correct use of t substitution and simplification

1 – Correct Integration

1 – final answer

(c) The line l_1 passes through the points $A(2,-1,4)$ and $B(4,-3,2)$.

(i) Show that a vector equation of line l_1 is $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. **2**

$\underset{\sim}{r} = k\overrightarrow{AB} + \overrightarrow{OA}$

$$\overrightarrow{OA} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 4-2 \\ -3-(-1) \\ 2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\underset{\sim}{r} = k(2)\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

1 – finding vector AB

1 – for establishing correct equation and relationships, using correct vector equations and notations.

(ii) The equation of line l_2 is given by $\frac{x+3}{k}=\frac{4-y}{6}=z-1$. **3**

Find the value of k for which l_1 and l_2 intersect.

Line 1	Line 2
$x=2+\lambda \quad --I$ $y=-1-\lambda \quad --II$ $z=4-\lambda \quad --III$	$\frac{x+3}{k}=\frac{4-y}{6} \quad --A$ $\frac{4-y}{6}=z-1 \quad --B$

Substitute Equations *II* and *III* into equation *B:*

$$\frac{4-(-1-\lambda)}{6}=4-\lambda-1$$

$$5+\lambda=18-6\lambda$$

$$\lambda=\frac{13}{7}$$

Substitute equations *I* and *II* into equation *A*:

$$\frac{2+\lambda+3}{k}=4-\lambda-1$$

$$5+\lambda=k(3-\lambda)$$

$$k=\frac{5+\lambda}{3-\lambda}$$

Substituting the value for λ

$$k=\frac{5+\frac{13}{7}}{3-\frac{13}{7}}$$

$$=6$$

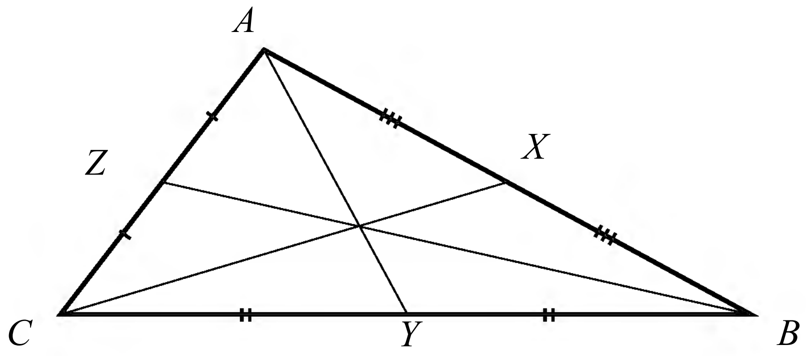
1 – Stablishing relationships for line 2

1 – equating the coordinates of the two lines and calculating lambda

1 – calculation of k.

(d) The diagram shows a triangle ABC.

The points X, Y and Z bisect the intervals AB, BC and CA respectively.



Show that $\overrightarrow{AY}+\overrightarrow{BZ}+\overrightarrow{CX}=0$ **3**

$$\overrightarrow{CA}+\overrightarrow{AB}+\overrightarrow{BC}=0 \qquad I$$

$$2\times\overrightarrow{CZ}+2\times\overrightarrow{AX}+2\times\overrightarrow{BY}=0$$

$$2\times\left(\overrightarrow{CZ}+\overrightarrow{AX}+\overrightarrow{BY}\right)=0$$

$$\left(\overrightarrow{CZ}+\overrightarrow{AX}+\overrightarrow{BY}\right)=0 \qquad II$$

$$\overrightarrow{AY}=\overrightarrow{CY}-\overrightarrow{CA}$$

$$\overrightarrow{BZ}=\overrightarrow{AZ}-\overrightarrow{AB}$$

$$\overrightarrow{CX}=\overrightarrow{BX}-\overrightarrow{BC}$$

By adding

$$\begin{aligned}\overrightarrow{AY}+\overrightarrow{BZ}+\overrightarrow{CX}&=\left(\overrightarrow{CY}-\overrightarrow{CA}\right)+\left(\overrightarrow{AZ}-\overrightarrow{AB}\right)+\left(\overrightarrow{BX}-\overrightarrow{BC}\right)\\&=\left(\overrightarrow{CY}+\overrightarrow{AZ}+\overrightarrow{BX}\right)-\left(\overrightarrow{CA}+\overrightarrow{AB}+\overrightarrow{BC}\right)\\&=0+0\\&=0\end{aligned}$$

Alternative Solution:

$$\overrightarrow{AY}=\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}$$

$$\overrightarrow{BZ}=\overrightarrow{BA}+\frac{1}{2}\overrightarrow{AC}$$

$$\overrightarrow{CX}=\overrightarrow{CB}+\frac{1}{2}\overrightarrow{BA}$$

$$\begin{aligned}\overrightarrow{AY}+\overrightarrow{BZ}+\overrightarrow{CX}&=\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CB}+\overrightarrow{BA}+\frac{1}{2}\overrightarrow{AC}+\overrightarrow{CB}+\frac{1}{2}\overrightarrow{BA}\\&=\frac{3}{2}(\overrightarrow{CB}+\overrightarrow{BA}+\overrightarrow{AC})\\&=\frac{3}{2}(0)\\&=0\end{aligned}$$

1 – Establishing base relations between required vectors and sides of the triangle which will lead to a solution

1 – Developing reasonable steps to lead to the answer

1 – complete solution.

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) If $x \le 1$ and $y \ge 1$ show that $x + y \ge 1 + xy$. **2**

$$x \le 1$$
$$x - 1 \le 0 \qquad I$$
$$y \ge 1$$
$$y - 1 \ge 0 \qquad II$$

$$(y-1)(x-1) \le 0 \qquad \text{From } I \text{ and } II$$
$$xy - x - y + 1 \le 0$$
$$xy + 1 \le x + y$$
$$x + y \ge xy + 1$$

1 - stablishing the relationships

2 - full solution

(b) Simplify fully $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$, given that ω is a non-real cube root of unity. **2**

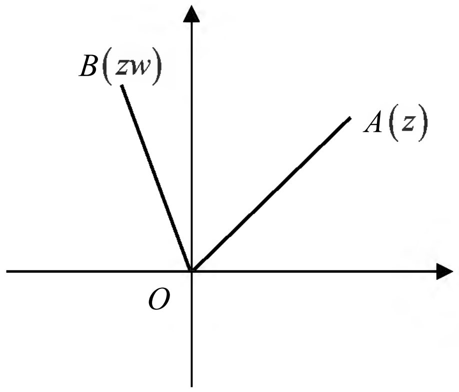
$$z^3 = 1$$
$$z^3 - 1 = 0$$
$$(z-1)(1+z+z^2) = 0$$
$$1 + \omega + \omega^2 = 0$$

$$\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = \frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$$
$$= \frac{1+\omega+\omega^2+1}{1+\omega+\omega^2+\omega^3}$$
$$= \frac{0+1}{0+\omega^3} \qquad \text{Since } 1+\omega+\omega^2 = 0$$
$$= \frac{1}{\omega^3} = 1 \qquad \text{Since } \omega^3 = 1$$

1 - stablishing the relationships

2 - full solution

(c) As shown on the Argand diagram below, the complex numbers z and zw are represented by the points A and B respectively.



Given $z = re^{i\theta}$ and $w = e^{i\frac{\pi}{3}}$, where $r > 0$,

(i) Explain why OAB is an equilateral triangle. **2**

- Multiplying by z by $e^{i\frac{\pi}{3}}$ rotates the vector represented by z anticlockwise through an angle of $\frac{\pi}{3}$.

  The length of z is unchanged as $|w| = 1$.
- Thus $|zw| = |z| = r$ and $\angle AOB = \frac{\pi}{3}$.

  Any isosceles triangle with the angle between the equal sides being $\frac{\pi}{3}$ is an equilateral triangle.

1 – stating reason for angle being 60 as well as length of sides

2 - full solution

(ii) Write the complex number $z - zw$ in exponential form in terms of r and θ. **2**

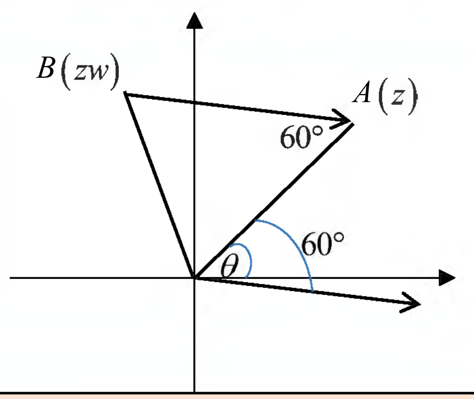
$z - zw = \overrightarrow{BA}$

$\arg(z - zw) = \frac{\pi}{3} - \theta$

$|BA| = |OB| = |OA| = r$ Equilateral triangle

$|z - zw| = r$

$z - zw = re^{i\left(\frac{\pi}{3} - \theta\right)}$



1 – answer in exponential form $re^{i\theta}$ with correct modulus;

1 – correct angle ($\frac{\pi}{3} - \theta$)

(d) (i) Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

$$RHS = \int_0^a f(a-x)\,dx$$
$$= \int_a^0 -f(u)\,du$$
$$= -\int_0^a -f(u)\,du$$
$$= \int_0^a f(x)\,dx$$
$$= LHS$$

Let	$u = a - x$	
	$du = -dx$	
When	$x = a$	$u = 0$
	$x = 0$	$u = a$

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x + \sin x}\,dx$. **3**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x + \sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right) + \sin\left(\frac{\pi}{4} - x\right)}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x + \sin x + \cos x - \sin x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{2\cos x}\,dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \tan x)\,dx$$
$$= \frac{1}{2}\left[x + \ln|\cos x|\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\right) - (0 + \ln 1)\right]$$
$$= \frac{1}{2}\left(\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\right)$$

1 – Simplifying the integral

1 – Integration

1 – Final answer

(e) A recurrence relation is defined by $u_{n+1} = \dfrac{3u_n - 1}{4u_n - 1}$ and $u_1 = 1$. **3**

Use Mathematical Induction to prove that $u_n = \dfrac{n}{2n-1}$ for $n \geq 1$.

Let $n = 1$

$LHS = u_n = 1$

$RHS = \dfrac{(1)}{2(1)-1} = \dfrac{1}{1} = 1$

True for $n = 1$

Assume true for $n = k$, where k is a positive integer.

$u_k = \dfrac{k}{2k-1}$

Prove true for $n = k+1$ if true $n = k$.

Aim: Prove that: $u_{k+1} = \dfrac{k+1}{2(k+1)-1}$

$u_{k+1} = \dfrac{k+1}{2k+1}$

$$\begin{aligned} LHS &= u_{n+1} \\ &= \frac{3u_k - 1}{4u_k - 1} \\ &= \frac{3\left(\frac{k}{2k-1}\right) - 1}{4\left(\frac{k}{2k-1}\right) - 1} \\ &= \frac{3k - (2k-1)}{4k - (2k-1)} \\ &= \frac{k+1}{2k+1} \\ &= RHS \end{aligned}$$

By Mathematical induction $u_n = \dfrac{n}{2n-1}$ is true for all integers, $n \geq 1$.

Question 15 (14 marks) Use a SEPARATE writing booklet

(a) (i) Show that $a^2 + 9b^2 \ge 6ab$, where a and b are real numbers. **1**

$$(a-3b)^2 \ge 0$$
$$a^2 - 6ab + 9b^2 \ge 0$$
$$a^2 + 9b^2 \ge 6ab$$

1 – correct setting out and result

(ii) Hence, or otherwise, show that $a^2 + 5b^2 + 9c^2 \ge 3(ab + bc + ac)$. **2**

$a^2 + 9b^2 \ge 6ab$ $\quad I$

$b^2 + 9c^2 \ge 6bc$ $\quad II$

$a^2 + 9c^2 \ge 6ac$ $\quad III$

By Adding I, II and III

$$a^2 + 9b^2 + b^2 + 9c^2 + a^2 + 9c^2 \ge 6ab + 6bc + 6ac$$
$$2a^2 + 10b^2 + 18c^2 \ge 6(ab + bc + ac)$$
$$a^2 + 5b^2 + 9c^2 \ge 3(ab + bc + ac)$$

1 – setting up the 3 inequalities or equivalent merit

1 – correct setting and result

(iii) Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$. **2**

$a > b > c > 0$

$ab > b^2 > bc > 0$

$ab > bc$ $\quad I$

$a > b > c > 0$

$ac > bc > c^2 > 0$

$ac > bc$ $\quad II$

$$\begin{aligned} a^2 + 5b^2 + 9c^2 &\ge 3(ab + bc + ac) && \text{From Part ii)} \\ &\ge 3(bc + bc + ac) && \text{From } I \\ &\ge 3(bc + bc + bc) && \text{From } II \\ &= 9bc \end{aligned}$$

$$\therefore a^2 + 5b^2 + 9c \ge 9bc$$

1 – using given condition to setup the 3 inequalities or equivalent merit

1 – correct setting and result

(b) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$. **4**

Use integration by parts to show that $I_n = \frac{2n}{2n+1} I_{n-1}$.

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}}\, dx$$

$$u = x^n \qquad v' = \frac{1}{\sqrt{1-x}} = (1-x)^{\frac{-1}{2}}$$

$$u' = nx^{n-1} \qquad v = -2(1-x)^{\frac{1}{2}} = -2\sqrt{1-x}$$

$$I_n = \left[-2nx^{n-1}\sqrt{1-x}\right]_0^1 - \int_0^1 -2nx^{n-1}\sqrt{1-x}\, dx$$

$$I_n = \left[\left(-2n(1)^{n-1}\sqrt{1-1}\right) - \left(-2n(0)^{n-1}\sqrt{1-0}\right)\right]_0^1 + 2n\int_0^1 \frac{x^{n-1}(1-x)}{\sqrt{1-x}}\, dx$$

$$I_n = 0 + 2n\int_0^1 \frac{x^{n-1} - x^n}{\sqrt{1-x}}\, dx$$

$$I_n = 2n(I_{n-1} - I_n)$$

$$I_n = 2nI_{n-1} - 2nI_n$$

$$(2n+1)I_n = 2nI_{n-1}$$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

1 – correct setup of parts

1 – correct algebraic manipulation

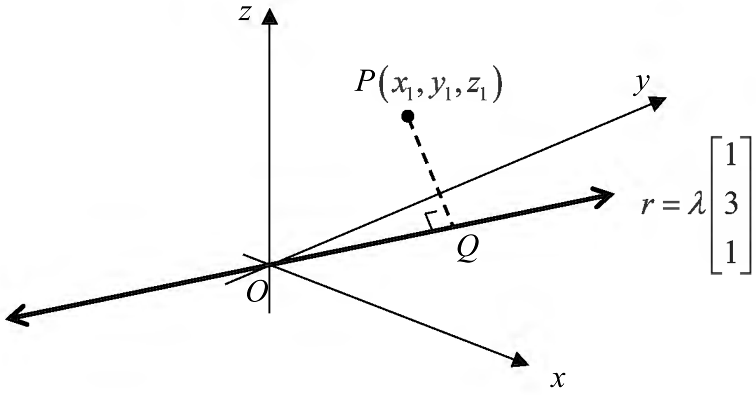
1 – correct integration

1 – correct answer

Question 15 Continues on Page 13

(c) The diagram below shows the line with vector equation $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

The point $P(x_1, y_1, z_1)$ is any point in the three-dimensional space.
Q is the point on $\underset{\sim}{r}$ such that PQ is a minimum.



(i) Using vector projection methods show that: **2**

$$\overrightarrow{OQ} = \frac{x_1 + 3y_1 + z_1}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$\overrightarrow{OQ}$ is the projection of $\overrightarrow{OP}$ onto line $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

Let $\overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ which is a vector on $\underset{\sim}{r}$

$$\overrightarrow{OQ} = proj_{\underset{\sim}{b}} \overrightarrow{OP}$$

$$= \frac{\overrightarrow{OP} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \times \underset{\sim}{b}$$

$$= \frac{(1)(x_1) + (3)(y_1) + (1)(z_1)}{1^2 + 3^2 + 1^2} \times \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

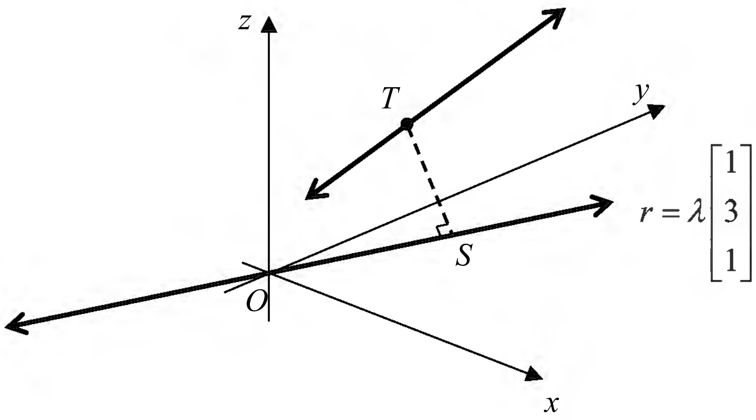
$$= \frac{x_1 + 3y_1 + z_1}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

1 – correct projection formula and application

1 – correct answer

It is given that line $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ does not intersect $\underset{\sim}{r}$.

Point T is on the line $\underset{\sim}{c}$ and S is a point on $\underset{\sim}{r}$ such that TS is a minimum.



(ii) By first expressing $\underset{\sim}{c}$ in parametric form, find an expression for the vector $\overrightarrow{TS}$ in terms of t. **3**

$\underset{\sim}{c} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ in parametric form is $\underset{\sim}{c} = \begin{bmatrix} 1-t \\ 4+t \\ 5t \end{bmatrix}$

If T is any point on $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ then the vector $\overrightarrow{OT}$ in terms of t is $\begin{bmatrix} 1-t \\ 4+t \\ 5t \end{bmatrix}$.

By substituting the parametric equation into the result in part i)

$$\overrightarrow{OS} = \frac{(1-t)+3(4+t)+(5t)}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\overrightarrow{OS} = \frac{13+7t}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\overrightarrow{TS} = \overrightarrow{OS} - \overrightarrow{OT} = \frac{13+7t}{11} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1-t \\ 4+t \\ 5t \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} (13+7t)-11(1-t) \\ 3(13+7t)-11(4+t) \\ (13+7t)-11(5t) \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 2+18t \\ -5+10t \\ 13-48t \end{bmatrix}$$

1 – correct parametric representation of T as a point on $\underset{\sim}{c}$

1 – correct calculation of $\overrightarrow{TS}$

1 – correct answer

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) (i) Prove that: **2**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\frac{\theta}{2}.$$

Let $t = \tan\frac{\theta}{2}$

$$\begin{aligned}
LHS &= \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} \\
&= \frac{\frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2} + 1} \\
&= \frac{1-t^2+2ti-1-t^2}{1-t^2+2ti+1+t^2} \\
&= \frac{2ti-2t^2}{2+2ti} \\
&= \frac{t(i-t)}{1+ti} \\
&= \frac{ti\left(1-\frac{t}{i}\right)}{1+ti} \\
&= \frac{ti(1+ti)}{1+ti} \\
&= it \\
&= i\tan\frac{\theta}{2}
\end{aligned}$$

(ii) Find the roots of the equation $w^5 = 1$. **2**
Express your answers in the form $(\cos\theta + i\sin\theta)$ where $-\pi < \theta \le \pi$.

$(\cos\theta + i\sin\theta)^5 = 1$

$(\cos 5\theta + i\sin 5\theta) = (\cos 2k\pi + i\sin 2k\pi)$ Where k is an integer.

$5\theta = 2k\pi$

$\theta = \frac{2k\pi}{5}$

$w = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ Where $k = 0, \pm1, \pm2$

(iii) Using parts (i) and (ii) find the roots of the equation: **3**

$$\left(\frac{2+z}{2-z}\right)^5 = 1$$

Since $w^5 = 1$

$$\frac{2+z}{2-z} = w$$

$$2+z = w(2-z)$$

$$2+z = 2w - zw$$

$$zw + z = 2w - 2$$

$$z(w+1) = 2(w-1)$$

$$z = \frac{2(w-1)}{w+1}$$

Since $w^5 = 1$ and from part (ii)

$w = \cos\theta + i\sin\theta$

$\theta = \frac{2k\pi}{5}$ Where $k = 0, \pm1, \pm2$

And from Part (i)

$$z = \frac{2(\cos\theta + i\sin\theta - 1)}{\cos\theta + i\sin\theta + 1}$$

$$= 2i\tan\frac{\theta}{2}$$

For $\theta = \frac{2k\pi}{5}$ Where $k = 0, \pm1, \pm2$

$z = 2i\tan\left(\frac{-2\pi}{5}\right)$, $2i\tan\left(\frac{-\pi}{5}\right)$, $2i\tan(0)$, $2i\tan\left(\frac{\pi}{5}\right)$, $2i\tan\left(\frac{2\pi}{5}\right)$

(b) (i) Show that $\dfrac{1}{(k-1)k(k+1)} \geq \dfrac{1}{k^3}$ where k is a positive integer greater than 1. **2**

(ii) Given that: **3**

$$S_n = \frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3} \quad \text{where } n \text{ is an integer and } n \geq 3.$$

$$\frac{1}{(k-1)k(k+1)} = \frac{A}{(k-1)}+\frac{B}{k}+\frac{C}{(k+1)}$$

$$1 = Ak(k+1)+B(k-1)(k+1)+C(k-1)k$$

$k=0$	$k=1$	$k=1$
$1=B(-1)(1)$	$1=A(1+1)$	$1=+C(-1-1)(-1)$
$B=-1$	$A=\dfrac{1}{2}$	$C=\dfrac{1}{2}$

$$\frac{1}{(k-1)k(k+1)} = \frac{1}{2(k-1)}-\frac{1}{k}+\frac{1}{2(k+1)}$$

$$\frac{1}{k^3} \leq \frac{1}{2(k-1)}-\frac{1}{k}+\frac{1}{2(k+1)}$$

$k=3$ $\quad \dfrac{1}{3^3} \leq \dfrac{1}{2\times 2}-\dfrac{1}{3}+\dfrac{1}{2\times 4}$

$k=4$ $\quad \dfrac{1}{4^3} \leq \dfrac{1}{2\times 3}-\dfrac{1}{4}+\dfrac{1}{2\times 5}$

$k=5$ $\quad \dfrac{1}{5^3} \leq \dfrac{1}{2\times 4}-\dfrac{1}{5}+\dfrac{1}{2\times 6}$

$k=6$ $\quad \dfrac{1}{6^3} \leq \dfrac{1}{2\times 5}-\dfrac{1}{6}+\dfrac{1}{2\times 7}$

$\downarrow \quad \downarrow \quad \downarrow$

$k=n-2$ $\quad \dfrac{1}{(n-2)^3} \leq \dfrac{1}{2\times(n-3)}-\dfrac{1}{(n-2)}+\dfrac{1}{2\times(n-1)}$

$k=n-1$ $\quad \dfrac{1}{(n-1)^3} \leq \dfrac{1}{2\times(n-2)}-\dfrac{1}{(n-1)}+\dfrac{1}{2\times n}$

$k=n$ $\quad \dfrac{1}{(n-1)^3} \leq \dfrac{1}{2\times(n-1)}-\dfrac{1}{n}+\dfrac{1}{2\times(n+1)}$

Using part (i) or otherwise, prove that $S_n < \dfrac{1}{12}$.

By adding and cancelling

$$S_n \le \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \frac{1}{2n} - \frac{1}{n} + \frac{1}{2n+2}$$

$$S_n \le \frac{1}{12} - \frac{1}{2n} + \frac{1}{2n+2}$$

$$S_n \le \frac{1}{12} + \frac{-n-1+n}{2n(n+1)}$$

$$S_n \le \frac{1}{12} + \frac{-1}{2n(n+1)}$$

$$S_n \le \frac{1}{12} \qquad \text{Since } \frac{-1}{2n(n+1)} < 0$$

(c) Find $\int \frac{\ln x - 1}{[x + \ln x]^2} dx$.

$$I = \int \frac{\ln x - 1}{[x + \ln x]^2} dx$$

$$= \int \frac{x + \ln x - x - 1}{[x + \ln x]^2} dx$$

$$= \int \frac{x + \ln x}{[x + \ln x]^2} dx - \int \frac{x+1}{[x + \ln x]^2} dx$$

$$= \int \frac{1}{x + \ln x} dx - \int \frac{x\left(1 + \frac{1}{x}\right)}{[x + \ln x]^2} dx$$

$$= \int \frac{1}{x + \ln x} dx - J$$

Let $J = \int \frac{x\left(1 + \frac{1}{x}\right)}{[x + \ln x]^2} dx$

By using integration by parts on J

$u = x \qquad v' = \frac{\left(1 + \frac{1}{x}\right)}{[x + \ln x]^2}$

$u' = 1 \qquad v = \frac{-1}{x + \ln x}$ by Reverse Chain Rule

$$J = x\left(\frac{-1}{x + \ln x}\right) - \int \frac{-1}{x + \ln x} dx$$

$$= \frac{-x}{x + \ln x} + \int \frac{1}{x + \ln x} dx$$

$$= \int \frac{1}{x + \ln x} dx - \left(\frac{-x}{x + \ln x} + \int \frac{1}{x + \ln x} dx\right)$$

$$= \frac{x}{x + \ln x} + C$$

Alternative Solutions
14 (a)

Alternative 1:

Assume $x + y < 1 + xy$

Then $xy - x - y + 1 > 0$

$x(y-1) - 1(y-1) > 0$

$(x-1)(y-1) > 0$

But $x - 1 \le 0 \quad (x \le 1)$

and $y - 1 \ge 0 \quad (y \ge 1)$

Therefore $(x-1)(y-1) \le 0$

Therefore, by contradiction, $x + y \ge 1 + xy$

Alternative 2:

$y \ge 1$ and $1 - x \ge 0$ (Given)

Therefore $y(1-x) \ge (1-x)$

$y - xy \ge 1 - x$

$x + y \ge xy + 1$

14 (b)

Alternative 1:

$$\frac{1}{1+\omega}+\frac{1}{1+\omega^2}=\frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$$

$$=\frac{2+\omega+\omega^2}{1+\omega+\omega^2+\omega^3}$$

$$=\frac{2+\omega+\omega^2}{1+\omega+\omega^2+1} \qquad \text{Since } \omega^3=1$$

$$=\frac{2+\omega+\omega^2}{2+\omega+\omega^2}$$

$$=1$$

Alternative 2:

$$\omega^3=1$$

$$\omega=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i \qquad \text{since } \omega \text{ is complex cube root of unity.}$$

$$\frac{1}{1+\omega}+\frac{1}{1+\omega^2}=\frac{1}{1-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i}+\frac{1}{1+\left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)^2}$$

$$=\frac{1}{\frac{1}{2}\pm\frac{\sqrt{3}}{2}i}+\frac{1}{1+\frac{1}{4}-\frac{3}{4}\mp\frac{\sqrt{3}}{2}i}$$

$$=\frac{2}{1\pm\sqrt{3}i}+\frac{2}{1\mp\sqrt{3}i}$$

$$=\frac{2(1-\sqrt{3}i)+2(1+\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} \quad \text{or} \quad \frac{2(1+\sqrt{3}i)+2(1-\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)}$$

$$=\frac{4}{1+3}$$

$$=1$$

14 (d) (i)

$$LHS = \int_0^a f(x)\,dx$$

$$= \int_a^0 -f(a-u)\,du$$

$$= -\int_0^a -f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx$$

$$= RHS$$

Let $x = a - u$

$dx = -du$

When $x = a$ $\quad u = 0$

$x = 0$ $\quad u = a$

End of paper